

Chiral Anomaly and $\gamma 3\pi^*$

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Abstract

Measurement of the $\gamma 3\pi$ process has revealed a possible conflict with what should be a solid prediction generated by the chiral anomaly. We show that inclusion of appropriate energy-momentum dependence in the matrix element reduces the discrepancy.

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1 Introduction

The chiral anomaly is a well-known and fascinating aspect of QCD. First identified in the context of the “triangle diagram” contribution to $\pi^0 \rightarrow 2\gamma$, [1] it has been shown to have much more general consequences which can be characterized in terms of an effective Lagrangian¹ [2]

$$\begin{aligned} \mathcal{L}_{WZW} = & \frac{N_c}{48\pi^2} \epsilon^{\mu\nu\alpha\beta} [e A_\mu \text{Tr}(Q(R_\nu R_\alpha R_\beta + L_\nu L_\alpha L_\beta)) \\ & - i e^2 F_{\mu\nu} A_\alpha \text{Tr}(Q^2(L_\beta + R_\beta) + \frac{1}{2}(QU^\dagger QUR_\beta + QUQU^\dagger L_\beta))] \end{aligned} \quad (1)$$

where $U = \exp(i \sum \lambda_i \phi_i / F_\pi)$ is the usual nonlinear matrix describing the pseudoscalar Goldstone fields, $R_\mu \equiv (\partial_\mu U^\dagger)U$, $L_\mu \equiv U(\partial_\mu U^\dagger)$ are right, left-handed currents respectively and $Q = \frac{e}{3}(2, -1, -1)_{\text{diag}}$ is the quark charge matrix. One immediately identifies the theoretical prediction for $\pi^0 \rightarrow \gamma\gamma$ which arises from the second line of Eq. 1

$$\begin{aligned} \text{Amp}_{\pi \rightarrow \gamma\gamma} &= -i A_{\gamma\gamma} \epsilon^{\mu\nu\alpha\beta} \epsilon_\mu^* k_\nu \epsilon_\alpha'^* k'_\beta \\ \text{with } A_{\gamma\gamma} &= \frac{\alpha N_c}{3\pi F_\pi} \xrightarrow{N_c=3} 0.025 \text{GeV}^{-1} \end{aligned} \quad (2)$$

which is in excellent agreement with the experimental value [3]

$$A_{\gamma\gamma} = 0.025 \pm 0.001 \text{GeV}^{-1} \quad (3)$$

In a corresponding fashion one can read off from Eq. 1 the prediction for the $\gamma\pi\pi\pi$ vertex

$$\begin{aligned} \text{Amp}_{\gamma\pi^+\pi^-\pi^0} &= -i A_{3\pi}(0) \epsilon^{\mu\nu\alpha\beta} \epsilon_\mu p_{1\nu} p_{2\alpha} p_{0\beta} \\ \text{with } A_{3\pi}(0) &= \frac{e N_c}{12\pi^2 F_\pi^3} \xrightarrow{N_c=3} 9.7 \text{GeV}^{-3} \end{aligned} \quad (4)$$

In this case, agreement with the value quoted experimentally [4]

$$A_{3\pi}^{\text{exp}} = 12.9 \pm 0.9 \pm 0.5 \text{GeV}^{-3} \quad (5)$$

is not particularly convincing and could even be said to favor the value $N_c = 4$! However, since such a violation would have severe consequences

¹We include here only the component relevant to electromagnetic interactions.

about the very foundations of QCD it warrants a more careful look, which is the purpose of the present note. Since the prediction of the anomaly strictly speaking hold only at zero four-momentum, while the experimental data is obtained over a range of energies above threshold, it is essential to understand the energy dependence of the $\gamma 3\pi$ amplitude generated by $\mathcal{O}(p^6)$ and higher contributions, and this is done in section II. Then in section III we use these results to confront existing experimental information and comment on implications for future experiments such as that approved at CEBAF.[5]

2 Finite Energy Corrections

The issue of finite energy correction to predictions of the anomaly has been addressed by a number of authors and is now reasonably well understood. The first such consideration was that of Terent'ev who, on phenomenological grounds, suggested the form[6]

$$A_{3\pi}(s, t, u) = A_{3\pi}(0)[1 + C_\rho e^{i\delta}(\frac{s}{m_\rho^2 - s} + \frac{t}{m_\rho^2 - t} + \frac{u}{m_\rho^2 - u})] \quad (6)$$

where $s = (p_1 + p_2)^2, t = (p_1 + p_0)^2, u = (p_2 + p_0)^2$, δ is an phenomenological phase factor, and

$$C_\rho = \frac{2g_{\rho\pi\pi}g_{\pi\rho\gamma}}{m_\rho^3 A_{3\pi}(0)} = 0.478 \quad (7)$$

represents the pure vector dominance contribution. The next step was taken by Rudaz who, noting that the amplitude for $\pi^0 \rightarrow \gamma\gamma$ could be generated entirely via the vector dominance diagram $\pi^0 \rightarrow \omega\rho \rightarrow \gamma\gamma$, *cf.* Figure 1a, proposed the same for the $\gamma 3\pi$ process, *cf.* Figure 1b, yielding[7]

$$A_{3\pi}(s, t, u) = \frac{1}{3}A_{3\pi}(0)[\frac{m_\rho^2}{m_\rho^2 - s} + \frac{m_\rho^2}{m_\rho^2 - t} + \frac{m_\rho^2}{m_\rho^2 - u}] \quad (8)$$

However, it was soon realized that this expression conflicted both with the KSRF relation[8] as well as with the anomalous Ward identities of Aviv and Zee[9] and that the correct form was[10]

$$A_{3\pi}(s, t, u) = -\frac{1}{2}A_{3\pi}(0)[1 - (\frac{m_\rho^2}{m_\rho^2 - s} + \frac{m_\rho^2}{m_\rho^2 - t} + \frac{m_\rho^2}{m_\rho^2 - u})] \quad (9)$$

Figure 1: Vector dominance contributions to the reactions $\pi^0 \rightarrow \gamma\gamma$ (a) and $\gamma \rightarrow 3\pi$ (b).

which contains both a vector dominance piece *and* a contact term.

In recent years, the problem has also been addressed via a one loop expansion in chiral perturbation theory, yielding the form, correct to $\mathcal{O}(p^6)$ in the derivative expansion[11]²

$$A_{3\pi}(s, t, u) = A_{3\pi}(0) \left[1 + \frac{3m_\pi^2}{2m_\rho^2} + \frac{m_\pi^2}{24\pi^2 F_\pi^2} \left(\frac{3}{4} \ln \frac{m_\rho^2}{m_\pi^2} + F(s) + F(t) + F(u) \right) \right] \quad (10)$$

where

$$F(s) = \begin{cases} \left(1 - \frac{s}{4m_\pi^2}\right) \sqrt{\frac{s-4m_\pi^2}{s}} \ln \frac{1 + \sqrt{\frac{s-4m_\pi^2}{s}}}{-1 + \sqrt{\frac{s-4m_\pi^2}{s}}} - 2 & s > 4m_\pi^2 \\ 2\left(1 - \frac{s}{4m_\pi^2}\right) \sqrt{\frac{4m_\pi^2-s}{s}} \tan^{-1} \sqrt{\frac{s}{4m_\pi^2-s}} - 2 & s \leq 4m_\pi^2 \end{cases} \quad (11)$$

The vector dominance form—Eq. 9—may be made consistent with its chiral counterpart—Eq. 10—provided we include the effects of final state p-wave pi-pi scattering. We begin by noting that the N/D form

$$t_1(s) = t_1^{CA}(s)/D_1(s), \quad (12)$$

²Here we use the mass shell condition $s + t + u = 3m_\pi^2$ and determine the coefficient of the term linear in s,t,u (a free parameter in strict chiral perturbation theory) by demanding agreement with expansion of the vector dominance form Eq. 9.

with

$$t_1^{CA}(s) = \frac{s - 4m_\pi^2}{96\pi F_\pi^2} \quad (13)$$

being the familiar p-wave Weinberg or current algebra prediction[12] and

$$D_1(s) = 1 - \frac{s}{m_\rho^2} - \frac{s}{96\pi^2 F_\pi^2} \ln \frac{m_\rho^2}{m_\pi^2} - \frac{m_\pi^2}{24\pi^2 F_\pi^2} F(s) \quad (14)$$

providing an analytic approximation to the Omnes function,³[13] provides a rather successful representation for the $\ell = 1$ pi-pi scattering amplitude[14]

$$t_1(s) = \sqrt{\frac{s}{s - 4m_\pi^2}} e^{i\delta_1(s)} \sin \delta_1(s). \quad (16)$$

Likewise, a reasonable approximation to the electromagnetic form factor of the charged pion is[15]

$$G_\pi(s) = 1/D_1(s) \approx \frac{m_\rho^2}{m_\rho^2 - s - im_\rho \Gamma_\rho(s)} \quad (17)$$

where

$$\Gamma_\rho(s) = \theta\left(\frac{s}{4m_\pi^2} - 1\right) \frac{g_{\rho\pi\pi}^2 s}{48\pi m_\rho} \left(1 - \frac{4m_\pi^2}{s}\right)^{\frac{3}{2}} \quad (18)$$

is an energy dependent quantity which reduces to the rho width when $s = m_\rho^2$. Here we have noted that

$$\frac{m_\pi^2}{24\pi^2 F_\pi^2} \text{Im} F(s) = \frac{1}{m_\rho} \Gamma_\rho(s) \quad (19)$$

and have utilized the KSFRF relation $g_{\rho\pi\pi}^2 = m_\rho^2/2F_\pi^2$. [8] We observe that Eqs. 9 and 10 can be made to agree to low order in s, t, u provided we use the

³One could also use the experimental p-wave phase shifts and the definition

$$D_1(s) = \exp\left(-\frac{s}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds' \delta_1(s')}{s'(s' - s - i\epsilon)}\right) \quad (15)$$

but the result is similar.

form

$$\begin{aligned}
A_{3\pi}(s, t, u) = & -\frac{1}{2}A_{3\pi}(0)\left[1 - \left(\frac{m_\rho^2}{m_\rho^2 - s} + \frac{m_\rho^2}{m_\rho^2 - t} + \frac{m_\rho^2}{m_\rho^2 - u}\right)\right] \\
& \times \left(\frac{1 - \frac{s}{m_\rho^2}}{D_1(s)}\right) \left(\frac{1 - \frac{t}{m_\rho^2}}{D_1(t)}\right) \left(\frac{1 - \frac{u}{m_\rho^2}}{D_1(u)}\right)
\end{aligned} \tag{20}$$

which is suggested by the feature that rescattering occurs in each of the three pi-pi channels simultaneously. It should also be noted that Eq. 20 satisfies the requirements of the Fermi-Watson theorem (*i.e.* unitarity) for the process $\gamma\pi \rightarrow \pi\pi$ and provides the preferred form to use in future analysis.

3 Comparison with Experiment

As mentioned in the introduction, it is often asserted that the experimental and theoretical values for $A_{3\pi}(0)$ are in significant disagreement. However, a more careful look at the paper of Antipov et al.[4] reveals that this is not the case. In fact, the experimental value quoted in Eq. 5 obtains only under the assumption that the matrix element $A_{3\pi}(s, t, u)$ is *independent* of momentum. On the other hand, averaging the from given by Terent'ev over the experimental spectrum yields (in units of GeV^{-3})[4]

$$A_{3\pi}^2(0) + 1.9 \cos \delta A_{3\pi}(0) + 1 = 166 \pm 23 \pm 13 \tag{21}$$

Since the spectral shape given by Terent'ev—Eq. 6—is basically in agreement with the form given by anomaly considerations—Eq. 9—provided $\cos \delta = 1$, and since the experiment of Antipov et al. was primarily at low values of the energy where unitarity corrections given by Eq. 19 are small we find the solution

$$A_{3\pi}(0) = 11.9 \pm 0.9 \pm 0.5 \text{GeV}^{-3} \tag{22}$$

Thus the disagreement with the number required by the chiral anomaly is at the 1.6σ level rather than the 2.3σ level generally quoted. Nevertheless, the experimental value is still on the high side and should certainly be subjected to additional experimental scrutiny, as will take place in the approved CLAS experiment at CEBAF.[5] When such data are analyzed they should use forms such as Eq. 20 which both satisfy chiral and unitarity restrictions

Table 1: Spectral modifications to the process $\gamma\pi \rightarrow \pi\pi$ generated via Eqs. 10,9,20 respectively. All values of s, t are in units of m_π^2 and the numbers quoted in the table represent percentage deviations from the anomaly prediction.

$s, -t$	0.5	1.0	2.0	3.0	4.0
4	6.0	6.0	6.0	6.3	6.6
	5.9	5.9	5.9	6.0	7.2
	6.9	6.9	6.9	8.5	10
5	6.6	6.6	6.6	6.7	6.8
	6.0	6.0	6.0	7.2	7.3
	8.5	8.5	8.5	10	11
10	8.7	8.7	8.7	8.6	8.5
	14	14	14	14	14
	21	21	21	21	21
15	11	11	11	11	11
	32	31	30	30	30
	45	45	44	44	44
20	14	14	14	14	14
	70	70	70	68	68
	96	95	95	93	93

as well as the phenomenological requirements of vector dominance. That use of such a form can make a significant difference can be seen in Table 1, where we compare the modifications of the lowest order anomaly prediction as generated by Eqs. 10,9,20.⁴ In the region $4m_\pi^2 < s < 13m_\pi^2; 0.5m_\pi^2 < |t| < 3.5m_\pi^2$ explored by the Antipov et al experiment the differences between the various forms are moderate, but in CEBAF proposal much larger values of energy and momentum transfer are involved— $4m_\pi^2 < s, |t| < 50m_\pi^2$ and the use of a properly unitarized form for the decay amplitude is essential in order to extract the value of the anomaly.

⁴The top line of each row is equivalent to the results quoted previously by Bijmens, Bramon and Cornet, ref 9.

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